# ESTIMABILITY STUDY OF THE PARAMETERS OF THE SEMI-ANALYTICAL LEE MODEL WITH HYPERSPECTRAL DATA

*Sicot Guillaume*$^\star$ *Ghannami Mohamed Ali*$^\star$ *Lennon Marc*$^\dagger$ *Loyer Sophie*$^\ddagger$

$^\star$ ENSTA Bretagne - M$^3$ Team - Lab-STICC, UMR-CNRS 6285, Brest, France
$^\dagger$Hytech-imaging, F-29200 Brest, France
$^\ddagger$Shom, Brest, France

## ABSTRACT

Hyperspectral sensors provide informative data in many fields related to Earth observation. On the coastal zone, the inversion of radiative transfer models of the light has shown the ability to estimate parameters characterizing the water column. Particularly the water column depth, its concentration of non-algal particles, in phytoplankton, and the bottom reflectance can be retrieved. In this paper, the ability of hyperspectral data to estimate the parameters of the coastal zone thanks to the semi-analytical Lee model is studied according to the way the measurement is made: number of bands, impact of spectral response at each band and without any additional concepts such as probabilistic concepts.

***Index Terms*—** estimability, radiative transfer model, shallow water, hyperspectral sensor design

## 1. INTRODUCTION

The estimation of water column parameters using spectral data can be performed thanks to two different types of methods, either by the so-called empirical method or by the inversion of a radiative transfer model. With hyperspectral data, this latter is the most commonly used. This approach consists in using a parametric model, a radiative transfer model to retrieve the best parameters according to data. In principle, this approach does not require ground truth data.

The radiative transfer model is therefore a key element in the estimation process of water column parameters. Classically, parameter estimation is assessed by considering the whole processing chain and the estimation quality is evaluated with ground truth as in [1]. In this study, the ability of the model to allow parameter estimation and more precisely the effect on the parameter estimation of the way the data are acquired is investigated thanks to the estimability function defined in [2]. In particular, the number of bands is studied as well as the fact that the measurements are not point measurements but rather integrated over spectral bands. Although we do not need to model measurement noise and therefore to introduce notions of probability, our approach can be adapted to take it into account (norm in the spectral domain, equation (3)) and could be related to studies such as the one presented in [3].

In this paper the radiative transfer model used is the semi-analytical Lee model [4], which is the most popular model in order to estimate water column parameters. This model is briefly introduced in section 2. The estimability function is introduced in section 3. This function is defined thanks to an optimization problem. As this problem is difficult to solve, an algorithm based on interval arithmetic is used. The principle of this optimization algorihtm is recalled in section 4. Finally the results of the estimability function are presented in a last section.

## 2. RADIATIVE TRANSFER MODEL

The quantity described by optical shallow water model is the remote sensing reflectance spectra, $R_{rs}(\lambda)$ and defined as

$$R_{rs}(\lambda) = \frac{L_u(\lambda)}{E_d(\lambda)},$$

where $L_u(\lambda)$, $E_d(\lambda)$ are respectively the upwelling radiance and the downwelling irradiance above the air/water interface. The variable $\lambda$ represents the wavelength. The first transformation deals with the interface air/water. The remote sensing reflectance spectra under the sea surface, $r_{rs}$, is calculated according to the expression hereafter [5]:

$$R_{rs} = \frac{\zeta r_{rs}}{1 - \Gamma r_{rs}}.$$

For numerical applications, $\zeta$ is set to 0.5 and $\Gamma$ to 1.5. The radiative transfer model used in this study is the semi-analytical model proposed by Lee [6] and defined as below:

$$r_{rs} = r_{rs}^{dp}\left(1 - e^{-(K_d + K_u^C)z}\right) + \frac{\rho}{\pi} e^{-(K_d + K_u^B)z}, \quad (1)$$

where $r_{rs}^{dp}$ is the deep water remote sensing spectrum, $K_d$ the diffuse attenuation coefficient of the downwelling irradiance, $K_u^C$ the diffuse attenuation coefficient for the upwelling radiance from the water scattering, $K_u^B$ the diffuse attenuation coefficient for the upwelling radiance from the bottom

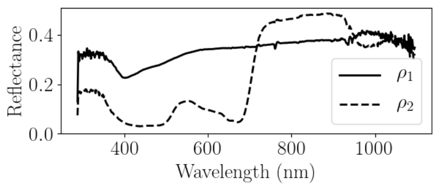



**Fig. 1**: Reflectance spectra, $\rho_1$ and $\rho_2$, according to wavelengths

reflectance, $\rho$ the bottom reflectance spectrum and $z$ the water column depth. The authors encourage the reader to refer to [4] to retrieve the expression of all these quantities.

The complete model used in this study will not be detailed in this paper. In particular, the expression of $r_{rs}^{dp}$, $K_d$, $K_u^C$ and $K_u^B$ of the equation (1) according to the absorption $a$ and the back-scattering $b_b$ will not be recalled. They are all based on the works from [7, 8, 9]. Hence only four parameters are considered in the model defined in (1): $C_{phy}$, the concentration of phytoplankton, $a^*_{cdom}(\lambda_0)$, the absorption due to colored dissolved organic matter at $\lambda_0$=400nm, $C_{nap}$, the concentration of non-algal particles and $z$, the water column depth..

Concerning the bottom reflectance, one uses the simplest model, where $\rho$ is a linear combination of two spectra. Hence $\rho$ can be written as follows.

$$\rho = \alpha\rho_1 + (1-\alpha)\rho_2$$

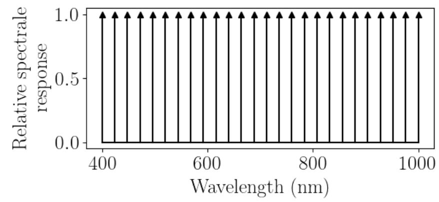
The spectrum $\rho_1$ (resp. $\rho_2$) is the reflectance spectrum of a white sand (resp. an alga) as shown on figure 1.

The unknown parameters for the general model used in this paper are $\boldsymbol{\theta} = \{C_{phy}, a^*_{cdom}(\lambda_0), C_{nap}, z, \alpha\}$. In this paper, the notation $r_{rs}(\lambda, \boldsymbol{\theta})$ means that the radiative transfer model is computed with the parameters $\boldsymbol{\theta}$ at the wavelength $\lambda$.

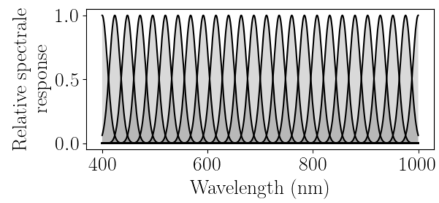
The issue addressed by the notion of estimability is the ability of retrieving all the parameters of $\boldsymbol{\theta}$ from a reflectance spectrum measurement. In other words, can one unambigously identify all the parameters of $\boldsymbol{\theta}$ thanks to the radiative transfer model. In this paper, the estimability property of the radiative transfer model is studied thanks to the estimability function defined in [2].

## 3. ESTIMABILITY FUNCTION

The notion of estimability, as defined in [2] associates two aspects of a measurement : the physical model and the design. In our context, the physical model is the radiative transfer model described in the previous section (section 2). The design describes how the measurement is performed. Each band used to construct the reflectance spectrum is the physical model observed through the sensor. Hence the sensor by its design is involved in the ability of a system to retrieve the model parameters.




(a) Design of a hyperspectral sensor as a Ddirac comb.




(b) Design of an hyperspectral sensor with 24nm FWHM bands (gaussian distribution).

**Fig. 2**: Examples of design for hyperspectral sensor. The spectral difference between central bands is set to 24 nm (26 bands).

The parameters $\boldsymbol{\theta}$ are said locally[1] estimable thanks to measurement $\boldsymbol{\eta}_S$ with the design $S$ if it verifies the following property ([2]).

$$\text{Let } \boldsymbol{\theta'} \in \mathcal{V}(\boldsymbol{\theta}), \text{ with } \mathcal{V}(\boldsymbol{\theta}), \text{ if } \boldsymbol{\eta}_S(\boldsymbol{\theta'}) = \boldsymbol{\eta}_S(\boldsymbol{\theta}) \Longrightarrow \boldsymbol{\theta'} = \boldsymbol{\theta}$$

Where $\mathcal{V}(\boldsymbol{\theta})$ is a neighborhood of $\boldsymbol{\theta}$. In [2], the authors propose to compute the estimability function, $E_{\boldsymbol{\eta}_S,\boldsymbol{\theta}}$ in order to analyze the parameters estimability. This function is defined as below.

$$E_{\boldsymbol{\eta}_S,\boldsymbol{\theta}}(\delta) = \min_{\{\boldsymbol{\theta'}:\|\boldsymbol{\theta'}-\boldsymbol{\theta}\|^2=\delta\}} \|\boldsymbol{\eta}_S(\boldsymbol{\theta'}) - \boldsymbol{\eta}_S(\boldsymbol{\theta})\|^2 \quad (2)$$

Considering the definition of the local estimability, if $\forall\delta \in \mathbb{R}^+ E_{\boldsymbol{\eta}_S,\boldsymbol{\theta}}(\delta) > 0$, one can say that $\boldsymbol{\theta}$ is estimable. This first result can be extended to a more quantitive analysis if one introduces the following parameter $\omega^*_{\boldsymbol{\eta}_S,\boldsymbol{\theta}} = \min_{\delta\in\mathbb{R}^+} \frac{E_{\boldsymbol{\eta}_S,\boldsymbol{\theta}}(\delta)}{\delta}$. This parameter brings some information on the location of the global minimizer [2]. It is desirable, in an estimability point of view, to get a high value for the parameter $\omega^*_{\boldsymbol{\eta}_S,\boldsymbol{\theta}}$. This last property is useful when one wants to compare design. Indeed the estimability function appears suitable, as it takes its value in the real set, to compare design of sensors. Hence a design $S_1$ is a priori more suitable than a design $S_2$ to estimate the parameter $\boldsymbol{\theta}$ if $\omega_{\boldsymbol{\eta}_{S_1},\boldsymbol{\theta}} > \omega_{\boldsymbol{\eta}_{S_2},\boldsymbol{\theta}}$. This is equivalent to verifying that $\forall\delta \in \mathbb{R}^+$, $E_{\boldsymbol{\eta}_{S_1},\boldsymbol{\theta}}(\delta) > E_{\boldsymbol{\eta}_{S_2},\boldsymbol{\theta}}(\delta)$. Due

[1] A global estimability is also defined in [2] but this property is not studied in this paper.

to numerical issues, in this paper, the square root of the estimability function, noted $\tilde{E}_{\boldsymbol{\eta}_S,\boldsymbol{\theta}}(\delta) = \sqrt{E_{\boldsymbol{\eta}_S,\boldsymbol{\theta}}(\delta^2)}$, is rather computed.

One needs to define the norms used in the equations (2). In the spectral space, the norm used is defined as below.

$$\|\boldsymbol{\eta}_S(\boldsymbol{\theta'}) - \boldsymbol{\eta}_S(\boldsymbol{\theta'})\|^2 = \frac{1}{N}\sum_{i=1}^{N}\left(\eta_{S,i}(\boldsymbol{\theta'}) - \eta_{S,i}(\boldsymbol{\theta'})\right)^2 \quad (3)$$

with $N$, the number of bands, and $\eta_{S,i}(\boldsymbol{\theta})$ and $\eta_{S,i}(\boldsymbol{\theta'})$ the component of the vector $\boldsymbol{\eta}_S(\boldsymbol{\theta})$ and $\boldsymbol{\eta}_S(\boldsymbol{\theta'})$ respectively. Concerning the parameter space the norm is defined as below

$$\|\boldsymbol{\theta'} - \boldsymbol{\theta}\|^2 = \sum_{i=1}^{N_{\boldsymbol{\theta}}}\left(\frac{\theta'_i - \theta_i}{\theta_i}\right)^2$$

with $N_{\boldsymbol{\theta}}$, the number of parameters in $\boldsymbol{\theta}$, *i.e.* $N_{\boldsymbol{\theta}} = 5$ in our context, and $\theta_i$ and $\theta'_i$ are the components of the vector $\boldsymbol{\theta}$ and $\boldsymbol{\theta'}$ respectively. One recalls that $\boldsymbol{\theta} = (C_{phy}, a^*_{cdom}(\lambda_0), C_{nap}, z, \alpha)$.

This norm is deliberately dependent on the parameter $\boldsymbol{\theta}$. Indeed the relative distance is more convenient to compare results among the different parametrizations.

In this paper, two kinds of design are considered.

- The sensor is considered perfect, it is able to measure perfectly the radiance at a specific wavelength, hence the design is a Dirac comb, see figure 2a for an example.
- In order to proceed to the measurement of the radiance at a specific wavelength, the sensor integrated the radiance around this specific radiance according to a gaussian distribution defined by its Full Width at Half Maximum (FWHM) see figure 2b for an example.

As the design is taken into account in the estimation process, the notations need to precise it. Let $N$ be the number of bands sampled by the hyperspectral sensor. When the design is a Dirac comb ($DC$), the set of all the reflectance bands is noted $\mathbf{R}_{rs,DC}(\boldsymbol{\theta})$ and is defined as

$$\mathbf{R}_{rs,DC}(\boldsymbol{\theta}) = \{R_{rs}(\lambda_1, \boldsymbol{\theta}), \ldots, R_{rs}(\lambda_N, \boldsymbol{\theta})\}$$

where the $\lambda_i$ indicate the sampled wavelengths and $R_{rs}(\lambda, \boldsymbol{\theta})$ is our physical model (see the previous section). For the Gaussian distribution design ($GD$), the set of all the reflectance bands is noted $\mathbf{R_{rs,GD}}(\boldsymbol{\theta})$ and is defined as below

$$\mathbf{R}_{rs,GD}(\boldsymbol{\theta}) = \{R_{rs,1}(\boldsymbol{\theta}), \ldots, R_{rs,N}(\boldsymbol{\theta})\}$$
$$\text{where } R_{rs,i}(\boldsymbol{\theta}) = \int_{\mathbb{R}^+} R_{rs}(\lambda, \boldsymbol{\theta}) f_\sigma(\lambda - \lambda_i)d\lambda$$

where $f_\sigma(\lambda)$ is the spectral response of each band. In this study $f_\sigma(\lambda)$ has the same expression as the density probability of a centered gaussian distribution with a standard deviation $\sigma = \frac{FWHM}{2\sqrt{2\ln 2}}$. The estimation stage aims to evaluate the parameter $\boldsymbol{\theta}$ from the measurements, which are collected into $\mathbf{R}_{rs,DC}(\boldsymbol{\theta})$ or $\mathbf{R}_{rs,GD}(\boldsymbol{\theta})$.

**Table 1**: Description of the set of parameters of Lee model studied in this paper

| Parameters | Set of values |
|---|---|
| **Radiative transfer model parameters** | |
| $C_{phy}$ (in mg.m$^{-3}$) | 1.0 |
| $a^*_{cdom}(\lambda_0)$ (in m$^{-1}$) | 0.1 |
| $C_{nap}$ (in g.m$^{-3}$) | 1.0 |
| $z$ (in m) | 1, 6, 12, 18 |
| $\alpha$ | 0.5 |
| **Hyperspectral sensor parameters** | |
| $N$ | 13, 25, 51, 101 |
| FWHM (in nm) | 6, 8, 12 |

This paper intends to compute the estimability function as defined in the equation (2) for a hyperspectral sensor. The optimization problem involved in the definition of the estimability function in the equation (2) is complicated. The complexity comes from the constrained set, *i.e.* $\Theta_{\boldsymbol{\theta},\delta} = \{\boldsymbol{\theta'} : \|\boldsymbol{\theta'} - \boldsymbol{\theta}\|^2 = \delta\}$ over which the optimization has to be done. In order to properly solve this optimization problem, an algorithm based on the interval arithmetic is used.

## 4. OPTIMIZATION ALGORITHM BASED ON INTERVAL ARITHMETIC

Although the optimization algorithm is not the main topic of this paper, some basic idea behind the interval arithmetic will be introduced. An interval is a closed connected subset of $\mathbb{R}$ [10]. Intervals are denoted thanks to square brackets: $[x]$. A non-empty interval $[x]$ can be represented by its endpoints:

$$[x] = [a, b] = \{x \,:\, a \leq x \leq b\}$$

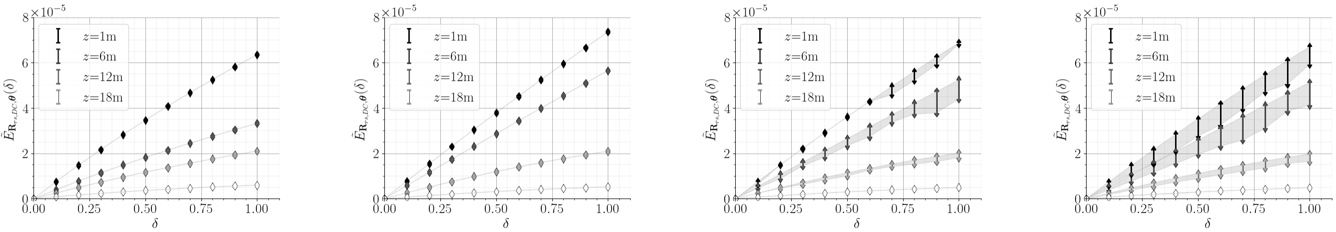
with $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$ and $a \leq b$. The set of intervals is denoted by $\mathbb{IR}$.

All the classical operations of arithmetic are defined over the intervals. For example, with $[x] = [a, b]$ and $[y] = [c, d]$, one has:

$$[x] + [y] = [a + c, b + d], \qquad [x] - [y] = [a - d, b - c]$$
$$[x] \cdot [y] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$$

These simple examples illustrate the approach induced by the interval arithmetic. The objective is to guarantee that the result of an operation belongs to an interval. This principle can be extended to complex and non-linear functions. In the Ibex library[2], interval arithmetic is associated with a branch

[2] http://www.ibex-lib.org/




(a) Estimability function for the 13-bands sensor

(b) Estimability function for the 25-bands sensor

(c) Estimability function for the 51-bands sensor

(d) Estimability function for the 101-bands sensor

**Fig. 3**: Estimability function for various water depth for a moderately turbid water (see table 1) for three configurations of hyperspectral sensors with a Dirac comb design.

and bound strategy in order to provide an optimizer for non-linear problems with inequality or equality constraints. This optimization algorithm thus provides the best solution found as well as an interval which contains the global optimum, which corresponds to the value of the estimability function.

In the following section both the optimal value obtained by the optimization algorithm and the interval containing the global optimum are presented. The use of optimizer based on the intervals guarantees that the global optimum is located within the yielded interval, leading to an analysis which is based on firm results.

## 5. RESULTS & DISCUSSION

In this section, two properties of the hysperspectral sensors are studied thanks to the estimability function : the number of bands $N$ and the width of the spectral response (the FWHM). As the estimability is studied locally, various parametrizations are defined as shown in the table 1. In this table, one can notice that only the water column depth is varying, the water constituents have been set to suit a moderately turbid water.

### 5.1. Analysis of the number of bands ($N$)

The number of bands, $N$, of the hypserspectral sensors is studied first. For this analysis, only the design with the Dirac comb is used. The configurations used are detailed below.

- A 13-band configuration between 400nm to 1000nm with a 50nm spectral period.
- A 25-band configuration with a 25nm spectral period.
- A 51-band configuration with a 12nm spectral period.
- A 101-band configuration with a 6nm spectral period.

The estimability function $\tilde{E}_{\mathbf{R}_{rs,DC},\boldsymbol{\theta}}(\delta)$ computed for the model parameter described in the table 1 is shown in the figures 3a, 3b, 3c and 3d. Before discussing these figures, it is useful to describe the representation used to show the results of the optimization algorithm. As described previously in section 4, the true value $\tilde{E}_{\mathbf{R}_{rs,DC},\boldsymbol{\theta}}(\delta)$ necessarily belongs to the interval provided by the optimization algorithm. These intervals are represented in the figures 3a, 3b, 3c and 3d by the error bars. Arbitrarily, between them, the error bars are linearly linked in order to provide more readable graphs. However the value of the estimability function $\tilde{E}_{\mathbf{R}_{rs,DC},\boldsymbol{\theta}}(\delta)$ is only guaranteed at the location of the error bars by themselves. It is noteworthy that the computation with a larger number of bands algorithmically is more cumbersome, this can be seen from the size of the error bars. Indeed the stopping criterion of the algorithm is only based on the computation time.

Firstly it can be noted that the estimability function is clearly dependent on the point of analysis. Whatever the number of bands, it seems that the parameters are more estimable when the water column depth $z = 1$m than when $z = 6$m and so on. Indeed $\tilde{E}_{\mathbf{R}_{rs,DC},\boldsymbol{\theta}}(\delta)$ is always greater for $z = 1$m, than for $z = 6$m for all $\delta$, than for $z = 12$m .... Although it seems that the order of the estimability function is monotonous with the water column depth, this statement is not true. Indeed, the estimability function when $z = 0.5$m is relatively similar to the one with $z = 6$m. The result with $z = 0.5$m is not represented on the figures 3 to prevent from overlapping between graphs and keep these figures readable.

Secondly, it appears that the number of bands does not affect significantly the estimability of the parameters of the Lee model for a number of bands greater than 25 bands. Indeed, the position of the estimability function seems identical within the precision of the optimization algorithm for 25, 51 and 101 bands. For a 13 bands sensor, the estimability function appears to have significantly lower values when the water column depth, $z = 1$m and more specifically when $z = 6$m. Even it is not shown in this paper, the decrease of the estimability function affects all the studied cases and is even more significant for smaller numbers of bands. Indeed, the estimability function clearly shows that the effect of the number of bands to retrieve the parameters of the Lee model is not linear and depends on the studied case: type of water depth when the number of bands is reduced. To go further, one should evaluate if the number of bands improves the robustness against measurements noise.

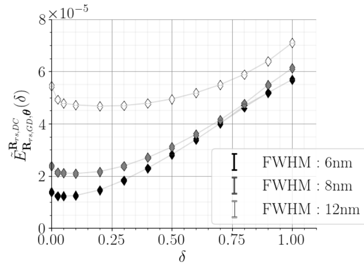



**Fig. 4**: Modified estimability function for a 51-bands configuration with a 6m water column depth and a moderately turbid water for a Dirac comb, when one considers the measurements provided by three different sensors with a 6nm, 8nm and a 12nm FWHM Gaussian distribution comb.

### 5.2. Analysis of the spectral response design

In this section, one wants to evaluate the influence of considering the design of a Dirac comb when the data is measured with a Gaussian distribution comb. For that purpose, one has slightly modified the estimability function as below

$$\tilde{E}^{\mathbf{R_{rs,DC}}}_{\mathbf{R_{rs,GD}},\boldsymbol{\theta}}(\delta) = \min_{\{\boldsymbol{\theta}' : \|\boldsymbol{\theta}'-\boldsymbol{\theta}\|=\delta\}} \|\mathbf{R_{rs,DC}}(\boldsymbol{\theta}') - \mathbf{R_{rs,GD}}(\boldsymbol{\theta})\|$$

The goal of this function is to evaluate the deviation made if one considers the model $R_{rs,DC}$ instead of $R_{rs,GD}$. The results are presented on the figure 4 for a 6m water column depth. The interpretation of this figure is slightly different in this context because it appears that $\tilde{E}^{\mathbf{R_{rs,DC}}}_{\mathbf{R_{rs,GD}},\boldsymbol{\theta}}(0) \neq 0$. It is noticeable that the minimum in the neighbourhood under consideration, when one considers the model $\mathbf{R_{rs,DC}}$ instead of the model $\mathbf{R_{rs,GD}}$ is not at the point chosen to build the spectrum. The minimum of the function $\tilde{E}^{\mathbf{R_{rs,DC}}}_{\mathbf{R_{rs,GD}},\boldsymbol{\theta}}(\delta)$ is not at $\delta = 0.0$. For a 6nm FWHM, it appears around $\delta = 0.05$, around $\delta = 0.1$ for a 8nm FWHM and around $\delta = 0.2$ for a 12nm FWHM. This suggest that a bias on the optimal parameters is introduced by neglecting the FWHM. For both cases it is essentially on the parameters of the water column (mainly on the $C_{phy}$ quantity, in this case, with for instance a bias of around 4.3% for the 6nm FWHM, 9% for the 8nm FWHM and 18.2% for the 12nm FWHM).

## 6. CONCLUSION

This study evaluates the ability of the semi-analytical Lee model to estimate water column parameters from hyperspectral data. In particular this paper studies the effect of how the radiance, and then the reflectance, is measured : number of bands, spectral response of the bands. The estimability function is the mathematical tool used to carry out this study.

Although, the study was restricted to the only case of moderately turbid water, it appears that the effect of the number of bands is not linear. For this type of water, the number of bands affects the estimability of the Lee model parameters when it is lesser than 25 bands. Moreover this limit seems to be dependent on the water column depth. This study also highlighted the fact that, not considering the spectral response of each band in the treatment, can introduce a bias in the estimation of the parameter. Obviously this bias is dependent on the width of the FWHM. It seems appropriate to continue this study with other types of water and configurations of acquisition to confirm these preliminary results.

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